Reliable computation of H infini central controllers near the optimum

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Rapports de Recherche

1992

25ème anniversaire

N° 1642

Programme 5
Traitement du Signal,
Automatique et Productique

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Mars 1992

RR-1642

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Abstract: The state-space formulas for the usual H_∞ central controller become singular when approaching the optimum γ_{opt}. A new approach is taken to circumvent this difficulty. It consists of extending the notion of central controller to include proper controllers with a feedthrough term. While such controllers are still derived from the usual Riccati solutions X_∞ and Y_∞, their feedthrough gain can be selected so as to neutralize the singularities near γ_{opt}. This provides numerically stable formulas for the controller parameters and eliminates the discontinuity between the realizations of nearly optimal and of reduced-order optimal central controllers. The advantages of this method are illustrated on a few examples.

Un Algorithme Fiable pour le Calcul de Contrôleurs H_∞ au Voisinage de l'Optimum

Résumé: Les formules classiques du contrôleur central H_∞ deviennent singulières lorsqu'on approche de l'optimum γ_{opt}. Une approche originale est présenté pour remédier à cette difficulté. On étend la notion de contrôleur central de façon à inclure les contrôleurs avec un terme proportionnel. Ces contrôleurs sont générés à partir des mêmes équations de Riccati mais le gain proportionnel permet d'absorber les singularités au voisinage de γ_{opt}. Ceci donne des formules numériquement stables et continues en γ_{opt}. En particulier, la réduction d'ordre résulte de simplifications pole/zéro finis et non infinis comme dans le cas du contrôleur central classique. Les avantages de cette approche sont illustrés sur quelques exemples.

1 Introduction

Doyle/Glover's state-space solution to the H_∞ suboptimal control problem [2, 7] has grown increasingly popular due to its easy implementation and numerical appeal. In this approach, solvability is characterized by three positivity conditions involving the solutions to a pair of indefinite Riccati equations. Explicit state-space formulas for some particular solution K_c called "central controller" are also provided.

When approaching the optimal H_∞ attenuation γ_{opt} however, singularities arise in these formulas which render computations sensitive. Indeed, the central controller is strictly proper but approaches a non strictly proper, reduced-order controller near γ_{opt}. Hence some pole/zero cancellation(s) must occur at infinity and this implies large magnitudes in the state-space parameters of K_c. The resulting realization is ill-conditioned and numerical spillover may cause performance degradation or even loss of internal stability (see Section 7). These difficulties are most acute at γ_{opt} where actual singularities require switching to a less convenient descriptor form [11] or extracting a reduced-order controller via alternative formulas [3, 8, 12].

The main concern of this paper is the elimination of numerical ill-condition not only at γ_{opt}, but also in its vicinity. As noticed before, ill-condition stems from the infinite pole/zero cancellations which occur near γ_{opt}. In turn, those result from the fact that suboptimal central controllers are strictly proper whereas their limit at γ_{opt} contains some feedthrough term. These observations suggest the following remedy: replace suboptimal K_c's with proper controllers which themselves contain a feedthrough term. We can then hope for finite pole/zero cancellations and smoothness of the realization parameters when approaching γ_{opt}. Interestingly, such alternative suboptimal controllers are easily constructed from the same Riccati solutions X_∞ and Y_∞. Moreover, the feedthrough term can indeed be chosen to desensitize the computation of the controller parameters. The efficacy of this method is best demonstrated at the optimum where it actually cancels singularities and smoothly leads to an explicit reduced-order realization of some optimal controller.

For simplicity, the results are established in the particular framework of the Standard Problem, although they have counterparts for more general H_∞ problems. The paper is organized as follows. Section 2 recalls the problem setup and formalism. Section 3 summarizes new results on the representation of H_∞ suboptimal controllers which prove useful to construct "generalized" central controllers with a feedthrough gain. Considering this extended class of central controllers, Section 4 shows that near singularities of $\gamma^{-2}X_\infty Y_\infty - I$ can be neutralized by appropriate choice of the feedthrough term. The selection of this term is formulated as a minimization problem and solutions are characterized in terms of elementary linear matrix equations. Section 5 discusses the optimal case and gives singularity-free formulas for optimal generalized central controllers. Smoothness between the suboptimal and optimal realizations is also established. Finally, numerical implementation of desensitized suboptimal and of generalized optimal central controllers is addressed in Section 6 and performance is evaluated on a few examples in Section 7.

2 Problem Setup and Motivations

Consider a plant

$$G(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (sI - A)^{-1} \begin{pmatrix} B_1 & B_2 \end{pmatrix} \qquad (2.1)$$

which maps exogenous inputs w and control inputs u to controlled outputs z and measured outputs y: $\begin{pmatrix} z \\ y \end{pmatrix} = G(s)\begin{pmatrix} w \\ u \end{pmatrix}$. Here z, y, w, and u are vectors of size p_1, p_2, m_1, and m_2, respectively, and $A \in \mathbf{R}^{n \times n}$. Moreover it is assumed that $m_1 \geq p_2$ and $p_1 \geq m_2$. When closing the plant G by a feedback law $u = K(s)y$, the closed-loop transfer function from w to z is given by the linear fractional map:

$$\mathcal{F}(G,K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}. \tag{2.2}$$

Given some attenuation γ, the H_∞ suboptimal control problem is as follows:

$$\mathcal{P}_\infty(\gamma)\text{: find an internally stabilizing controller } K(s) \text{ such that } \|\mathcal{F}(G,K)\|_\infty \leq \gamma \tag{2.3}$$

The optimal H_∞ attenuation (or gain) γ_{opt} is defined as the smallest γ for which $\mathcal{P}_\infty(\gamma)$ is solvable. As mentioned earlier, the discussion is specialized to the Standard Problem (SP) [2] and the following assumptions are therefore made:

(A1) (A, B_2, C_2) is stabilizable and detectable.

(A2) $D_{11} = D_{22} = 0$.

(A3) D_{12} and D_{21} are full rank, $D_{12}^T(D_{12}, C_1) = (I, 0)$, and $D_{21}(D_{21}^T, B_1^T) = (I, 0)$.

(A4) $rank\begin{pmatrix} j\omega I - A & -B_2 \\ C_1 & D_{12} \end{pmatrix} = n + m_2$ and $rank\begin{pmatrix} j\omega I - A & B_1 \\ -C_2 & D_{21} \end{pmatrix} = n + p_2$ for all $\omega \in \mathbf{R}$, or equivalently (A, B_1, C_1) has no uncontrollable or unobservable mode on the imaginary axis.

The SP is not fully general as observed in [4] but has the merit of simplifying calculations and also the situation at the optimum (see below). For more details on assumptions (A1)-(A4), see, e.g., [7, 6].

In [2], solvability of $\mathcal{P}_\infty(\gamma)$ for $\gamma > \gamma_{opt}$ is characterized by the following conditions:

(C1) There exists stabilizing solutions X_∞ and Y_∞ to the game Riccati equations (GRE):

$$A^TX + XA + X(\gamma^{-2}B_1B_1^T - B_2B_2^T)X + C_1^TC_1 = 0\,; \qquad AY + YA^T + Y(\gamma^{-2}C_1^TC_1 - C_2^TC_2)Y + B_1B_1^T = 0. \tag{2.4}$$

(C2) These solutions moreover satisfy $X_\infty \geq 0$, $Y_\infty \geq 0$, and $\rho(X_\infty Y_\infty) < \gamma^2$.

The solutions X_∞ and Y_∞ are also used to construct a particular solution of $\mathcal{P}_\infty(\gamma)$ called central controller and defined as $K_c := C_c(sI - A_c)^{-1}B_c$ with

$$\begin{aligned} A_c &= A + (\gamma^{-2}B_1B_1^T - B_2B_2^T)X_\infty - (I - \gamma^{-2}Y_\infty X_\infty)^{-1}Y_\infty C_2^TC_2; \\ B_c &= (I - \gamma^{-2}Y_\infty X_\infty)^{-1}Y_\infty C_2^T\,; \qquad C_c = -B_2^TX_\infty. \end{aligned} \tag{2.5}$$

Note that X_∞, Y_∞, and K_c are functions of γ.

As γ reaches γ_{opt}, some of conditions (C1)-(C2) fail(s). As a special feature of the SP, the conditions $X_\infty \geq 0$ and $Y_\infty \geq 0$ cannot fail prior to $\rho(X_\infty Y_\infty) < \gamma^2$ [4]. Consequently, γ_{opt} corresponds to one of the following two situations:

- X_∞ or Y_∞ is no longer stabilizing, that is, one of the Hamiltonian matrices associated with the GRE's (2.4) has eigenvalue(s) on the imaginary axis.

- $\rho(X_\infty Y_\infty) = \gamma_{opt}^2$.

In the first case, $X_\infty, Y_\infty, A_c, B_c$, and C_c all have finite limits at γ_{opt} and the difficulty is easily circumvented. By contrast, if $I - \gamma^{-2} Y_\infty X_\infty$ becomes singular at γ_{opt} then A_c and B_c defined by (2.5) become unbounded.

Theoretically, a remedy consists of rewriting the realization (2.6) of K_c in the following equivalent descriptor form [11]:

$$(I - \gamma^{-2} Y_\infty X_\infty)\dot{\xi} = F\xi + Y_\infty C_2^T y; \qquad u = -B_2^T X_\infty \xi \tag{2.6}$$

where $F = (I - \gamma^{-2} Y_\infty X_\infty)(A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty) - Y_\infty C_2^T C_2$. It can be shown that this descriptor system is always of index one [8] and defines an optimal reduced-order controller where the unbounded modes of A_c have cancelled at infinity. Formulas for this controller appear in [3, 8, 12]. Note that their applicability requires exact singularity of $I - \gamma^{-2} Y_\infty X_\infty$. Consequently, there is a region near γ_{opt} where no numerically sound formula is available. By contrast, the state-space formulas of the generalized central controllers introduced in the sequel behave smoothly near the optimum and are insensitive to ill-conditioning of $(I - \gamma^{-2} Y_\infty X_\infty)^{-1}$.

3 Generalized Central Controllers

In [5], a family of controllers solving $\mathcal{P}_\infty(\gamma)$ is described in terms of parametric GREs. The GREs (2.4) are a special case of these parametric equations and they characterize the central controller K_c when imposing strict properness. If this constraint is relaxed however, (2.4) yields a larger family of proper suboptimal controllers within which numerically-sound alternatives to K_c can be sought near γ_{opt}. We begin by recalling a constructive result of [5].

Theorem 3.1 *Assume (A1)-(A4) and consider any matrices X, Y in $\mathbf{R}^{n \times n}$, M, N in $\mathbf{R}^{n \times r}$ $(r \le n)$, and B_K, C_K, D_K which jointly satisfy:*

(i) *$D_K \in \mathbf{R}^{m_2 \times p_2}$ and $\sigma_{max}(D_K) \le \gamma$.*

(ii) *M, N are full column rank.*

(iii) *X and Y are stabilizing solutions to the GREs*

$$\begin{aligned} &A^T X + XA + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X + C_1^T C_1 + \\ &\qquad (XB_2 + MC_K^T + C_2^T D_K^T)(I - \gamma^{-2} D_K D_K^T)^{-1}(XB_2 + MC_K^T + C_2^T D_K^T)^T = 0; \end{aligned} \tag{3.1}$$

$$\begin{aligned} &AY + YA^T + Y(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y + B_1 B_1^T + \\ &\qquad (YC_2^T + NB_K + B_2 D_K)(I - \gamma^{-2} D_K^T D_K)^{-1}(YC_2^T + NB_K + B_2 D_K)^T = 0, \end{aligned} \tag{3.2}$$

with the coupling condition

$$MN^T = \gamma^{-2} XY - I. \tag{3.3}$$

(iv) *$X \ge 0$, $Y \ge 0$, and $\rho(XY) \le \gamma^2$.*

Then if A_K denote the (unique) solution to

$$\begin{aligned} NA_K M^T = &- \{A + B_2 D_K C_2 + \gamma^{-2}(YA^T X + B_1 B_1^T X + YC_1^T C_1) + B_2 C_K M^T + NB_K C_2\} - \\ &(YC_2^T + B_2 D_K + NB_K)(I - \gamma^{-2} D_K^T D_K)^{-1} D_K^T (B_2^T X + D_K C_2 + C_K M^T), \end{aligned} \tag{3.4}$$

the r-th order controller $K(s) := D_K + C_K(sI - A_K)^{-1} B_K$ solves $\mathcal{P}_\infty(\gamma)$. ∎

For $\gamma > \gamma_{opt}$, the usual central controller K_c is obtained by chosing $X, Y, M, N, B_K, C_K, D_K$ of Theorem 3.1 so that

(a) $XB_2 + MC_K^T + C_2^T D_K^T = 0$ and $YC_2^T + NB_K + B_2 D_K = 0$.

(b) $D_K = 0$.

Item (a) ensures that $X = X_\infty$ and $Y = Y_\infty$ and amounts to minimizing the Riccati solutions X and Y among all possible solutions compatible with **(i)-(iv)**. This extremality property accounts for the role played by the central controller equations in H_∞ theory (see [5] for details). By contrast, item (b) is immaterial and only ensures that the controller is strictly proper. This requirement is now relaxed to generalize the notion of central controller as follows.

Definition 3.2 *We call generalized central controller any solution $K(s) = D_K + C_K(sI - A_K)^{-1}B_K$ of $\mathcal{P}_\infty(\gamma)$ obtained by setting*

$$XB_2 + MC_K^T + C_2^T D_K^T = 0; \qquad\qquad YC_2^T + NB_K + B_2 D_K = 0; \tag{3.5}$$

in (3.1)-(3.2) and selecting $X = X_\infty$ and $Y = Y_\infty$.

Note that Theorem 3.1 guarantees that any such K solves $\mathcal{P}_\infty(\gamma)$. Also, equation (3.4) for the controller state matrix A_K simplifies for generalized central controllers to [5]:

$$\begin{aligned} NA_K M^T &= \left\{A + Y_\infty(\gamma^{-2}C_1^T C_1 - C_2^T C_2)\right\}(\gamma^{-2}Y_\infty X_\infty - I) - B_2 C_K M^T \\ &= (\gamma^{-2}Y_\infty X_\infty - I)\left\{A + (\gamma^{-2}B_1 B_1^T - B_2 B_2^T)X_\infty\right\} - NB_K C_2. \end{aligned} \tag{3.6}$$

For $\gamma > \gamma_{opt}$, the feedthrough gain D_K is arbitrary provided that $\sigma_{max}(D_K) \le \gamma$. In fact, the set of generalized central controllers can be parametrized by D_K upon rewriting (3.5) and (3.6) as:

$$NB_K = -(Y_\infty C_2^T + B_2 D_K); \tag{3.7}$$

$$C_K M^T = -(B_2^T X_\infty + D_K C_2); \tag{3.8}$$

$$NA_K M^T = \{A + Y_\infty(\gamma^{-2}C_1^T C_1 - C_2^T C_2)\}(\gamma^{-2}Y_\infty X_\infty - I) + B_2(B_2^T X_\infty + D_K C_2). \tag{3.9}$$

From (3.3) M and N are $n \times n$ nonsingular matrices for $\gamma > \gamma_{opt}$ so that A_K, B_K, C_K are entirely determined once D_K is chosen. Observe that the resulting controller is of order n and independent of the particular choice of M, N compatible with (3.3). Indeed, changing M, N simply amounts to a change of state coordinate in the controller realization. Since D_K is free within the ball of radius γ, it can be utilized to desensitize the formulas (3.9)-(3.8) to near singularities in $\gamma^{-2}X_\infty Y_\infty - I$. This idea is developed in Section 4.

Finally, the optimal case $\gamma = \gamma_{opt}$ is a little trickier due to the singularity of $\gamma_{opt}^{-2}X_\infty Y_\infty - I$. Indeed, M and N are then nonsquare so that the choice of D_K is constrained through (3.7)-(3.8). Section 6 shows that adequate D_K's are then obtained by solving a Parrott problem and that the resulting generalized central controller is of order $r < n$ where $n - r$ is the rank deficiency of $\gamma_{opt}^{-2}X_\infty Y_\infty - I$.

4 Alleviating Ill-Condition near the Optimum

Generalized central controllers offer additional degrees of freedom via the feedthrough gain D_K. This section shows that ill-conditioning of $(I-\gamma^{-2}Y_\infty X_\infty)^{-1}$ near γ_{opt} can be neutralized by appropriate choice of D_K. To isolate singularities arising near γ_{opt}, split the singular value decomposition (SVD) of $\gamma^{-2}X_\infty Y_\infty - I$ into two parts:

$$\gamma^{-2}X_\infty Y_\infty - I = U_1\Sigma_1 V_1^T + U_2\Sigma_2 V_2^T \tag{4.1}$$

where

- Σ_1 and Σ_2 are diagonal and Σ_2 gathers the r smallest singular values,
- the columns of U_2 (V_2) are the left (right) singular vectors associated with the r smallest singular values,
- the columns of U_1 (V_1) are the left (right) singular vectors associated with the remaining singular values.

Recall that (U_1, U_2) and (V_1, V_2) are orthogonal matrices. Note also that the gap between the largest singular value of Σ_2 and the smallest of Σ_1 need not be quantified, although Σ_2 will typically gather those singular values which approach zero near γ_{opt}. Accordingly, U_2 and V_2 will contain the directions of "near singularity" of $\gamma^{-2}X_\infty Y_\infty - I$. Finally, recalling that M and N are only subject to (3.3), we select them as $M := (U_1\Sigma_1^{1/2}, U_2\Sigma_2^{1/2})$ and $N := (V_1\Sigma_1^{1/2}, V_2\Sigma_2^{1/2})$.

When the singular values of Σ_2 approach zero, M and N become singular and the inversions required to solve (3.9)-(3.8) are ill-conditioned. Yet, this does not necessarily result in large entries in A_K, B_K, C_K. Consider for instance (3.7) and let $\Pi := Y_\infty C_2^T + B_2 D_K$. The solution is $B_K = -\begin{pmatrix} \Sigma_1^{-1/2}V_1^T\Pi \\ \Sigma_2^{-1/2}V_2^T\Pi \end{pmatrix}$. From the SVD partitioning, a large magnitude can only stem from the term $\Sigma_2^{-1/2}V_2^T\Pi$. Now if Π is chosen to make $\|V_2^T\Pi\|$ small enough, the magnitude amplification associated with $\Sigma_2^{-1/2}$ can be totally offset. In such event, the term $\Sigma_2^{-1/2}V_2^T\Pi$ remains below the problem scale and its inaccuracy is unlikely to throw off performance and internal stability. Based on this observation, the computation of A_K, B_K, C_K will be least sensitive when the norms of

- $V_2^T(Y_\infty C_2^T + B_2 D_K)$ and $(B_2^T X_\infty + D_K C_2)U_2$,
- $\{(A + Y_\infty(\gamma^{-2}C_1^T C_1 - C_2^T C_2))(\gamma^{-2}Y_\infty X_\infty - I) + B_2(B_2^T X_\infty + D_K C_2)\}\, U_2$,
- $V_2^T\, \{(\gamma^{-2}Y_\infty X_\infty - I)(A + (\gamma^{-2}B_1 B_1^T - B_2 B_2^T)X_\infty) + (Y_\infty C_2^T + B_2 D_K)C_2\}$

are minimal. Recalling that $\|(\gamma^{-2}Y_\infty X_\infty - I)U_2\| = \|V_2^T(\gamma^{-2}Y_\infty X_\infty - I)\| = \|\Sigma_2\|$, it is easily seen that the last two expressions are small in norm whenever the first two are.

Hence, a reasonable objective for sensitivity reduction is to select D_K through the following constrained minimization problem:

$$\min_{D_K} C(D_K) \text{ subject to } \|D_K\|_F \le \gamma \tag{4.2}$$

where

$$C(D_K) := \|(B_2^T X_\infty + D_K C_2)U_2\|_F^2 + \|V_2^T(Y_\infty C_2^T + B_2 D_K)\|_F^2. \tag{4.3}$$

The Frobenius norm is chosen for tractability reasons. Note that $\|D_K\|_F \le \gamma$ is sufficient to enforce the constraint $\sigma_{max}(D_K) \le \gamma$ since $\|M\|_F \ge \sigma_{max}(M)$ for any matrix M. Recalling that $\|M\|_F^2 = trace(MM^T) = trace(M^TM)$, the Lagrangian associated with (4.2) is

$$\mathcal{L}(D_K, 2\alpha) = trace\left\{(B_2^TX_\infty + D_KC_2)U_2U_2^T(B_2^TX_\infty + D_KC_2)^T + (Y_\infty C_2^T + B_2D_K)^TV_2V_2^T(Y_\infty C_2^T + B_2D_K)\right\} + 2\alpha\left\{trace(D_KD_K^T) - \gamma^2\right\}. \tag{4.4}$$

With the notation

$$\Phi := B_2^TV_2V_2^TB_2; \qquad \Psi := C_2U_2U_2^TC_2^T; \qquad \Theta := B_2^T(X_\infty U_2U_2^T + V_2V_2^TY_\infty)C_2^T, \tag{4.5}$$

minimizers of this quadratic form in D_K are characterized by the first-order condition:

$$\frac{\partial \mathcal{L}}{\partial D_K} = (\Phi + \alpha I)D_K + D_K(\Psi + \alpha I) + \Theta \doteq 0. \tag{4.6}$$

The Sylvester equation (4.6) has a unique solution $D_K(\alpha)$ for all $\alpha > 0$. Observing that $\alpha = 0$ yields unconstrained minimizers of $C(D_K)$ while $\lim_{\alpha\to+\infty} D_K(\alpha) = 0$, the Lagrange multiplier α is a simple means of trade-off between optimality and constraint enforcement. Indeed, some $\alpha_0 \ge 0$ such that $\sigma_{max}(D_K(\alpha_0)) \le \gamma$ is easily obtained by line search. A desensitized controller is then derived using (3.9)-(3.8) with $D_K := D_K(\alpha_0)$. Interestingly, and to the extent of our current experience, the norm constraint on D_K was always satisfied by minimum-norm unconstrained ($\alpha = 0$) minimizers of $C(D_K)$ (provided that Σ_2 is associated with distinctively small singular values of $\gamma^{-2}X_\infty Y_\infty - I$). Despite its empirical nature, this fact justifies a closer look at the unconstrained problem. In addition, minimum-norm unconstrained minimizers of $C(D_K)$ turn out to be instrumental to optimal controller design (see next section). The next theorem characterizes solutions of the unconstrained problem.

Theorem 4.1 *Consider SVDs of the symmetric matrices Φ and Ψ:*

$$\Phi = W_1\Lambda_1^2W_1^T; \qquad \Psi = Z_1\Omega_1^2Z_1^T \tag{4.7}$$

where Λ_1, Ω_1 are diagonal nonsingular and $W = (W_1, W_2)$ and $Z = (Z_1, Z_2)$ are orthogonal. Then all unconstrained minimizers of $C(D_K)$ defined in (4.3) are of the form

$$D_K(\Delta) = W\begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta \end{pmatrix}Z^T \tag{4.8}$$

where Δ is arbitrary and $\Delta_{11}, \Delta_{12}, \Delta_{21}$ are uniquely determined by:

$$\Lambda_1^2\Delta_{11} + \Delta_{11}\Omega_1^2 + W_1^T\Theta Z_1 = 0; \tag{4.9}$$

$$\Lambda_1^2\Delta_{12} + W_1^T\Theta Z_2 = 0; \tag{4.10}$$

$$\Delta_{21}\Omega_1^2 + W_2^T\Theta Z_1 = 0. \tag{4.11}$$

Hence, minimum-norm minimizers are obtained by solving for Δ the following Parrott problem:

$$\min_{\Delta} \sigma_{max}\begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta \end{pmatrix}. \tag{4.12}$$

Proof: Unconstrained minimizers of $C(D_K)$ solve (4.6) for $\alpha = 0$. Pre- and post-multiply this equation by W^T and Z, respectively, and define $\Delta_{ij} := W_i^T D_K Z_j \; : i,j = 1,2$. This yields the equations (4.8)-(4.11) and $W_2^T \Theta Z_2 = 0$. The first three determine $\Delta_{11}, \Delta_{12}, \Delta_{21}$ uniquely while the last equation is trivial since $W_2^T B_2^T V_2 = 0$ and $U_2^T C_2^T Z_2 = 0$. The block Δ_{22} is therefore arbitrary and the other statements follow.

■

Recall that the smallest achievable value in the minimization (4.12) is [10, 11]:

$$\sigma^* := \max \left\{ \sigma_{max}(\Delta_{11}, \Delta_{12}), \sigma_{max} \left(\begin{array}{c} \Delta_{11} \\ \Delta_{21} \end{array} \right) \right\} \tag{4.13}$$

which is attained, in particular, for $\Delta^* = -\Delta_{21}(\sigma^{*2} I - \Delta_{11}^T \Delta_{11})^{-1} \Delta_{11}^T \Delta_{12}$ (provided that $\sigma_{max}(\Delta_{11}) < \sigma^*$). As mentioned earlier, $D_K(\Delta^*)$ was always found to satisfy the norm constraint $\sigma_{max}(D_K) \le \gamma$ and was therefore chosen as feedthrough term to achieve the best sensitivity reduction. Interestingly, this is also an appropriate choice for the optimal case $\gamma = \gamma_{opt}$, as shown in the next section. Finally, experiments confirm that near singularities in $\gamma^{-2}XY - I$ are indeed cancelled with this approach so that their effect on the computed values of A_K, B_K, C_K is negligible (see Section 8).

5 Generalized Central Controllers at the Optimum

We restrict our attention to those cases - the most common - where the optimum is characterized by the failure of the condition $\rho(X_\infty Y_\infty) < \gamma^2$. We therefore assume that the GREs (2.4) retain stabilizing solutions X_∞ and Y_∞ at γ_{opt} and that $\rho(X_\infty Y_\infty) = \gamma_{opt}^2$. In such cases, the usual central controller K_c tends toward a non strictly proper reduced-order controller in the frequency domain [8, 12]. The state-space formulas (2.5) are then singular at γ_{opt}. By contrast, the formulas for sensitivity-reducing controllers introduced in Section 4 **smoothly** extend to the optimal case and involve finite instead of infinite pole/zero cancellations at γ_{opt}. Hence, such controllers offer a numerically stable alternative to K_c near γ_{opt}. Note that the smooth behavior at γ_{opt} is hardly surprising since these controllers have been designed precisely to cancel singularities of $\gamma_{opt}^{-2} X_\infty Y_\infty - I$. Finally, it is worth mentioning that the limit of K_c at γ_{opt} is not among optimal generalized central controllers in general (see Remark 5.3 below).

In virtue of Theorem 3.1 and Definition 3.2, $K(s) = D_K + C_K(sI - A_K)^{-1} B_K$ is an optimal generalized central controller if and only if B_K, C_K, D_K satisfy

$$\sigma_{max}(D_K) \le \gamma_{opt} \qquad \text{and} \qquad \left\{ \begin{array}{l} X_\infty B_2 + M C_K^T + C_2^T D_K^T = 0 \; ; \\ Y_\infty C_2^T + N B_K + B_2 D_K = 0 \end{array} \right. \tag{5.1}$$

where X_∞, Y_∞ denote the stabilizing solutions of (2.4) and $MN^T = \gamma_{opt}^{-2} X_\infty Y_\infty - I$. Note that the order of K is equal to the rank of $\gamma_{opt}^{-2} X_\infty Y_\infty - I$. Introduce the counterpart of the SVD (4.1):

$$\gamma_{opt}^{-2} X_\infty Y_\infty - I = U_1 \Sigma_1 V_1^T; \qquad \Sigma_1 > 0. \tag{5.2}$$

Since $U_2^T M = 0$ and $N^T V_2 = 0$, D_K is now actively constrained by the two equations in (5.1) and must satisfy:

$$\sigma_{max}(D_K) \le \gamma_{opt} \qquad \text{and} \qquad \left\{ \begin{array}{l} D_K(C_2 U_2) + B_2^T X_\infty U_2 = 0 \; ; \\ (B_2^T V_2)^T D_K + (C_2 Y_\infty V_2)^T = 0 \, . \end{array} \right. \tag{5.3}$$

Conversely, for any such D_K the system (5.1) is solvable for B_K, C_K and an optimal controller can be reconstructed via Theorem 3.1. Hence, optimal generalized central controllers are parametrized by all feedthrough D_K which satisfy (5.3). In the sequel, such D_K's are referred to as *γ_{opt}-admissible.*

The following identities are useful toward the characterization of γ_{opt}-admissible D_K's.

Lemma 5.1 *Let U_2 and V_2 denote orthonormal bases of the left and right kernels of $\gamma_{opt}^{-2}X_\infty Y_\infty - I$. Then the following identities hold at γ_{opt}:*

$$(C_2 Y_\infty V_2)^T (C_2 Y_\infty V_2) = \gamma_{opt}^2 \, (B_2^T V_2)^T (B_2^T V_2); \tag{5.4}$$

$$(B_2^T X_\infty U_2)^T (B_2^T X_\infty U_2) = \gamma_{opt}^2 \, (C_2 U_2)^T (C_2 U_2); \tag{5.5}$$

$$(C_2 Y_\infty V_2)^T (C_2 U_2) = (B_2^T V_2)^T (B_2^T X_\infty U_2). \tag{5.6}$$

Consequently, we have SVDs of the form:

$$B_2^T V_2 = W_1 \Lambda_1 \Xi^T; \qquad C_2 Y_\infty V_2 = \gamma_{opt} \, P_1 \Lambda_1 \Xi^T; \qquad C_2 U_2 = Z_1 \Omega_1 \Upsilon^T; \qquad B_2^T X_\infty U_2 = \gamma_{opt} \, Q_1 \Omega_1 \Upsilon^T \tag{5.7}$$

where Λ_1 and Ω_1 are diagonal positive definite matrices and the columns of W_1, P_1, Z_1, Q_1, Ξ, and Υ form orthonormal systems of vectors.

Proof: See Appendix A.

■

The next theorem establishes the existence of γ_{opt}-admissible feedthrough gains and characterizes them all.

Theorem 5.2 *Assume condition (C1) of Section 2 holds at γ_{opt}, consider the SVDs (5.7) of Lemma 5.1. and introduce W_2 and Z_2 such that $W = (W_1, W_2)$ and $Z = (Z_1, Z_2)$ are orthogonal matrices. With this assumption and notation,*

- *there always exist γ_{opt}-admissible feedthrough gains and they are all obtained as:*

$$D_K(\Delta_m) = -\gamma_{opt} \; W \begin{pmatrix} W_1^T Q_1 & P_1^T Z_2 \\ W_2^T Q_1 & \Delta_m \end{pmatrix} Z^T \tag{5.8}$$

  where Δ_m denotes any minimizer of the Parrott problem

$$\min_{\Delta} \; \sigma_{max} \begin{pmatrix} W_1^T Q_1 & P_1^T Z_2 \\ W_2^T Q_1 & \Delta \end{pmatrix}. \tag{5.9}$$

- *Any γ_{opt}-admissible feedthrough gain D_K satisfies $\sigma_{max}(D_K) = \gamma_{opt}$.*

Proof: Using the SVD's (5.7), the two equations in (5.3) reduce to

$$D_K Z_1 + \gamma_{opt} \, Q_1 = 0; \qquad W_1^T D_K + \gamma_{opt} \, P_1^T = 0. \tag{5.10}$$

Defining $\Delta_{ij} := W_i^T D_K Z_j; i, j = 1, 2$, (5.10) is equivalent to

$$\begin{pmatrix} \Delta_{11} \\ \Delta_{21} \end{pmatrix} = -\gamma_{opt} \begin{pmatrix} W_1^T Q_1 \\ W_2^T Q_1 \end{pmatrix}; \qquad (\Delta_{11}, \Delta_{12}) = -\gamma_{opt} \, (P_1^T Z_1, P_1^T Z_2). \tag{5.11}$$

Observe that $P_1^T Z_1 = W_1^T Q_1$ in virtue of (5.6) together with (5.7). Hence the two expressions of Δ_{11} in (5.11) are compatible and the solution set of (5.3) consists of all D_K's of the form (5.8) such that $\sigma_{max}(D_K) \leq \gamma_{opt}$.

Now, it is known from Parrott's theorem [10] that for any Δ_m,

$$\sigma_{max}(D_K(\Delta_m)) \geq \gamma_{opt} \max \left\{ \sigma_{max}(P_1^T Z_1, P_1^T Z_2), \sigma_{max} \begin{pmatrix} W_1^T Q_1 \\ W_2^T Q_1 \end{pmatrix} \right\}$$

and here the maximum evaluates to 1 since $\sigma_{max}(P_1^T Z_1, P_1^T Z_2) = \sigma_{max}(P_1^T Z) = \sigma_{max}(P_1) = 1$ and similarly $\sigma_{max} \begin{pmatrix} W_1^T Q_1 \\ W_2^T Q_1 \end{pmatrix} = 1$. Consequently, the constraint $\sigma_{max}(D_K) \leq \gamma_{opt}$ is satisfied if and only if Δ_m is chosen as a minimizer of (5.9), in which case we actually have $\sigma_{max}(D_K(\Delta_m)) = \gamma_{opt}$. ■

Remark 5.3 In general, the feedthrough gain D_{K_c} obtained from the descriptor formulas (2.6) of K_c at γ_{opt} [8, 12, 3] is not among the γ_{opt}-admissible D_K's of Theorem 5.2. Indeed, D_{K_c} is given as $D_{K_c} = -B_2^T X_\infty U_2 (C_2 U_2)^+$ which is a particular solution of the top equation in (5.3) but not necessarily of the bottom one. To be convinced, observe that with the notation (5.7), an alternative expression is $D_{K_c} = -\gamma_{opt}\, Q_1 Z_1^T$ so that $W_1^T D_{K_c} + \gamma_{opt}\, P_1^T = -\gamma_{opt} W_1^T Q_1 Z_1^T + \gamma_{opt} P_1^T = \gamma_{opt} P_1^T (I - Z_1 Z_1^T)$. Thus, the counterpart in (5.10) of the bottom equation in (5.3) is not satisfied unless $P_1^T (I - Z_1 Z_1^T) = P_1^T Z_2 Z_2^T = 0$, that is, unless $P_1^T Z_2 = 0$. Consequently, the limit of K_c at γ_{opt} is not among the generalized central controllers of Definition 3.2.

■

Theorem 5.2 provides a simple and numerically sound way of computing reduced-order optimal controllers when γ_{opt} is characterized by $\rho(X_\infty Y_\infty) = \gamma_{opt}^2$. Note that the final order of the controller explicitly appears throughout the design (rank of $\gamma_{opt}^{-2} X_\infty Y_\infty - I$, column dimensions of M and N, dimensions of A_K). Practical implementation of this scheme is discussed in the next section. To conclude this section, observe that the feedthrough gains of optimal generalized central controllers as characterized in Theorem 5.2 coincide with the minimum-norm unconstrained minimizers at γ_{opt} of $C(D_K)$ defined by (4.3). Hence there is a nice numerical continuity between the sensitivity-reducing controllers of Section 4 and the optimal generalized central controllers described in this section.

Theorem 5.4 *Let U_2 and V_2 denote the left and right null spaces of $\gamma_{opt}^{-2} X_\infty Y_\infty - I$. Then the γ_{opt}-admissible feedthrough gains D_K are exactly the minimum-norm solutions to the unconstrained problem:*

$$\min_{D_K} \left\{ \|(B_2^T X_\infty + D_K C_2) U_2\|_F^2 + \|V_2^T (Y_\infty C_2^T + B_2 D_K)\|_F^2 \right\}. \tag{5.12}$$

Proof: First observe that the criterion $\mathcal{L}(D_K) = \|(B_2^T X_\infty + D_K C_2) U_2\|_F^2 + \|V_2^T (Y_\infty C_2 + B_2 D_K)\|_F^2$ is always nonnegative and that $\mathcal{L}(D_K) = 0$ for any γ_{opt}-admissible feedthrough gain D_K based on (5.3). Consequently, γ_{opt}-admissible feedthrough gains are minimizers of $\mathcal{L}(D_K)$.

Conversely, unconstrained minimizers of (5.12) satisfy the first-order equation (4.6) with $\alpha = 0$. Using the SVDs (5.7), this equation can be rewritten

$$(W_1 \Lambda_1^2 W_1^T)\, D_K \;+\; D_K\, (Z_1 \Omega_1^2 Z_1^T) \;+\; \gamma_{opt}\, (Q_1 \Omega_1^2 Z_1^T + W_1 \Lambda_1^2 P_1^T) \;=\; 0.$$

With the notation $\Delta_{ij} := W_i^T D_K Z_j; i,j = 1,2$, pre- and post-multiplication by W^T and Z yields

$$\begin{aligned}\Lambda_1^2\Delta_{11} + \Delta_{11}\Omega_1^2 + \gamma_{opt}\,(W_1^T Q_1)\Omega_1^2 + \gamma_{opt}\,\Lambda_1^2(P_1^T Z_1) &= 0;\\ \Lambda_1^2\Delta_{12} + \gamma_{opt}\,\Lambda_1^2 P_1^T Z_2 &= 0;\\ \Delta_{21}\Omega_1^2 + \gamma_{opt}\,W_2^T Q_1\Omega_1^2 &= 0.\end{aligned}$$

From the last two equations it follows that $\Delta_{12} = -\gamma_{opt}\,P_1^T Z_2$ and $\Delta_{21} = -\gamma_{opt}\,W_2^T Q_1$. Meanwhile, the identity $P_1^T Z_1 = W_1^T Q_1$ (see proof of Theorem 5.2) allows to rewrite the first equation as

$$\Lambda_1^2(\Delta_{11} + \gamma_{opt}\,W_1^T Q_1) + (\Delta_{11} + \gamma_{opt}\,W_1^T Q_1)\Omega_1^2 = 0$$

which together with $\Lambda_1^2 > 0$ and $\Omega_1^2 > 0$ ensures that $\Delta_{11} + \gamma_{opt}\,W_1^T Q_1 = 0$. Consequently, any solution of (5.12) is of the form $D_K(\Delta)$ defined in (5.8) and minimum-norm solutions are obtained for minimizers Δ_m of the Parrott problem (5.9).

■

6 Numerical Implementation

Since the computation of X_∞ and Y_∞ is potentially ill-conditioned near γ_{opt}, it is advisable to implement modified controller formulas where X_∞ and Y_∞ are replaced with orthonormal bases $\begin{pmatrix} P_X \\ Q_X \end{pmatrix}$ and $\begin{pmatrix} P_Y \\ Q_Y \end{pmatrix}$ of the stable invariant subspaces of the corresponding Hamiltonian matrices:

$$H_\infty = \begin{pmatrix} A & \gamma^{-2}B_1B_1^T - B_2B_2^T \\ -C_1^TC_1 & -A^T \end{pmatrix}; \qquad J_\infty = \begin{pmatrix} A^T & \gamma^{-2}C_1^TC_1 - C_2^TC_2 \\ -B_1B_1^T & -A \end{pmatrix}. \tag{6.1}$$

Note that

$$H_\infty \begin{pmatrix} P_X \\ Q_X \end{pmatrix} = \begin{pmatrix} P_X \\ Q_X \end{pmatrix} T_H; \qquad J_\infty \begin{pmatrix} P_Y \\ Q_Y \end{pmatrix} = \begin{pmatrix} P_Y \\ Q_Y \end{pmatrix} T_J \tag{6.2}$$

where T_H and T_J have their spectrum in the open left-half plane, and that $X_\infty = Q_X P_X^{-1}$ and $Y_\infty = Q_Y P_Y^{-1}$. Consistently with Section 5, we assume that these subspaces remain of dimension n at γ_{opt}. Finally, such orthonormal bases can be computed in a numerically stable way via the Schur algorithm for Riccati equations [13, 9].

The results of Sections 4 and 5 have immediate counterparts in terms of P_X, Q_X, P_Y, Q_Y which are now summarized in the form of two algorithms.

Algorithm 6.1

Objective: Computation of generalized central controllers which are insensitive to ill-condition of $(I - \gamma^{-2}X_\infty Y_\infty)^{-1}$ near γ_{opt}.

1. *Compute the SVD of* $P_X^T P_Y - \gamma^{-2}Q_X^T Q_Y$ *and partition it as* $U_1\Sigma_1V_1^T + U_2\Sigma_2V_2^T$ *where* Σ_2 *gathers the distinctively small singular values.*

2. *Starting with* $\alpha = 0$ *and until the 2-norm of the solution* $D_K(\alpha)$ *is less than* γ, *solve for increasing values of* α *the Sylvester equation* $\Phi D_K(\alpha) + D_K(\alpha)\Psi + \Theta = 0$ *where*

$$\Phi := B_2^T P_Y V_2 V_2^T P_Y^T B_2 + \alpha I; \quad \Psi := C_2 P_X U_2 U_2^T P_X^T C_2^T + \alpha I; \quad \Theta := B_2^T\,(Q_X U_2 U_2^T P_X^T + P_Y V_2 V_2^T Q_Y^T)\,C_2^T. \tag{6.3}$$

In the particular case $\alpha = 0$, *compute* $D_K(\alpha)$ *as indicated in Theorem 4.1 with* Φ, Ψ, *and* Θ *given by (6.3).*

3. *Once an adequate $D_K(\alpha)$ has been found, set $D_K := D_K(\alpha)$ and compute B_K, C_K, A_K as the (unique) solutions of*

$$\Sigma^{1/2} B_K = V^T(Q_Y^T C_2^T + P_Y^T B_2 D_K); \qquad C_K \Sigma^{1/2} = (B_2^T Q_X + D_K C_2 P_X)U. \tag{6.4}$$
$$(V\Sigma^{1/2}) A_K (\Sigma^{1/2}U^T) = (\gamma^{-2}Q_X^T Q_Y - P_X^T P_Y)T_H + (Q_Y^T C_2^T + P_Y^T B_2 D_K)C_2 P_X. \tag{6.5}$$

Note that T_H is a direct by-product of the Schur decomposition (cf. (6.2)).

4. *The controller $K(s) := D_K + C_K(sI - A_K)^{-1}B_K$ is then a solution of $\mathcal{P}_\infty(\gamma)$.*

∎

Note that the Sylvester equation is derived from the modified criterion $\|(B_2^T Q_X + D_K C_2 P_X)U_2\|_F^2 + \|V_2^T(Q_Y^T C_2 + P_Y^T B_2 D_K)\|_F^2$ while (6.5) follows from the identity $\left\{A + (\gamma^{-2}B_1 B_1^T - B_2 B_2^T)X_\infty\right\} P_X = P_X T_H$. A standard algorithm to solve Sylvester equations is found in [1].

Algorithm 6.2

Objective: Computation of optimal generalized central controllers ($\gamma = \gamma_{opt}$) provided that H_∞ and J_∞ have no imaginary axis eigenvalue at γ_{opt}.

1. *Compute the SVD $P_X^T P_Y - \gamma_{opt}^{-2} Q_X^T Q_Y = U_1 \Sigma_1 V_1^T$ where $\Sigma_1 > 0$, and let U_2, V_2 be the orthogonal complements of U_1, V_1 generated by the SVD algorithm.*

2. *Paralleling (5.7), compute the SVDs*

$$B_2^T P_Y V_2 = W_1 \Lambda_1 \Xi^T; \quad C_2 Q_Y V_2 = \gamma_{opt} P_1 \Lambda_1 \Xi^T; \quad C_2 P_X U_2 = Z_1 \Omega_1 \Upsilon^T; \quad B_2^T Q_X U_2 = \gamma_{opt} Q_1 \Omega_1 \Upsilon^T. \tag{6.6}$$

3. *Find a minimizer Δ_m for the Parrott problem (5.9) (a possible solution is given at the end of Section 4) and set $D_K := D_K(\Delta_m)$.*

4. *Compute B_K, C_K, A_K as the (unique) solutions of*

$$\Sigma_1^{1/2} B_K = V_1^T(Q_Y^T C_2^T + P_Y^T B_2 D_K); \qquad C_K \Sigma_1^{1/2} = (B_2^T Q_X + D_K C_2 P_X)U_1. \tag{6.7}$$
$$(V_1\Sigma_1^{1/2}) A_K (\Sigma_1^{1/2}U_1^T) = (\gamma^{-2}Q_X^T Q_Y - P_X^T P_Y)T_H + (Q_Y^T C_2^T + P_Y^T B_2 D_K)C_2 P_X. \tag{6.8}$$

5. *Then $K(s) := D_K + C_K(sI - A_K)^{-1}B_K$ is a reduced-order H_∞ optimal controller.*

∎

As long as $\alpha = 0$ produces an admissible D_K in step 4 of Algorithm 6.1, both algorithms use the same constructive scheme: find a minimum-norm unconstrained minimizer of (5.12) and deduce A_K, B_K, C_K by solving (3.9)-(3.8). This ensures a smooth transition from one algorithm to the other when approaching γ_{opt}. In practise, the two algorithms are used concurrently and selected via some thresholding device on the smallest singular value of $\Gamma = P_X^T P_Y - \gamma^{-2}Q_X^T Q_Y$. Below the threshold, Γ is considered singular "at the machine precision" and K is designed with Algorithm 6.2. Otherwise, Algorithm 6.1 is utilized so that adverse effects associated with small singular values of Γ are minimized. The continuity near γ_{opt} of the computed state-space parameters A_K, B_K, C_K, D_K ensures a smooth transition between algorithms which is confirmed in practise.

The combination of Algorithms 6.1 and 6.2 performs well compared to existing methods as illustrated in the next section. Yet, numerical stability could further benefit from more analytical insight into the transition between nonsingularity and singularity of Γ. Directions of future research include showing that $\alpha = 0$ is always a satisfactory choice in Algorithm 6.1; finding *a priori* estimates of an adequate threshold for switching algorithm; reducing sensitivity with respect to singular values which are small but nonzero at γ_{opt} (possibly by exploiting the remaining degrees of freedom in D_K after solving the Parrott problem (5.9)).

7 Numerical Tests

Near γ_{opt}, Algorithms 6.1 and 6.2 offer substantial gain in numerical reliability over classical central controller implementations. This is illustrated by two examples, the first of which has a mostly didactical value while the second is quite representative of the general situation. For fair comparison, the formulas (2.5) for K_c are replaced with their counterpart in [11] where X_∞ and Y_∞ are not explicitly computed (cf. Section 6). Note that (6.4)-(6.5) coincide with the formulas in [11] when D_K is set to zero. All computations were performed in double-precision arithmetic on a Sun SPARCstation 2 with relative machine precision $\epsilon_m = 4.4\,10^{-16}$.

Example 7.1 Consider the Standard Problem associated with the data:

$$A = 1; \qquad B_1 = C_1^T = (1,0); \qquad B_2 = C_2 = 1; \qquad D_{12} = D_{21}^T = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad (7.1)$$

The optimal gain for this problem is $\gamma_{opt} \approx 2.732051$. For values of γ approaching γ_{opt}, the performance of K_c as implemented in [11] are compared to that of the generalized central controller returned by Algorithms 6.1 and 6.2. Three criterion of evaluation are considered:

- $Mmax.$: the magnitude of the largest entry encountered in the computed state-space parameters A_K, B_K, C_K of the controller,
- $\|\mathcal{F}(G,K)\|_\infty$: the actual closed-loop performance,
- $Smar.$: the relative internal stability margin as measured by the ratio $r(A_{cl})/\|A_{cl}\|$ where $A_{cl} := \begin{pmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{pmatrix}$ is the closed-loop state matrix and $r(.)$ denotes the (complex) stability radius of a matrix [14].

These values as well as the feedthrough term D_K obtained with the generalized central controller approach appear in Table 7.2.

	Algorithms 6.1-6.2				Usual algorithm		
γ	D_K	$\|\mathcal{F}(G,K)\|_\infty$	$Mmax.$	$Smar.$	$\|\mathcal{F}(G,K)\|_\infty$	$Mmax.$	$Smar.$
3.0	−2.67	2.77	2.7	0.6	2.99	$1.4\ 10^1$	$6\ 10^{-2}$
2.8	−2.72	2.74	2.7	0.6	2.80	$4.7\ 10^1$	$3\ 10^{-2}$
2.75	−2.73	2.74	2.7	0.6	2.75	$1.6\ 10^2$	$9\ 10^{-3}$
2.735	−2.731	2.732	2.7	0.6	2.735	$1.0\ 10^3$	$1\ 10^{-3}$
2.7325	−2.7319	2.7321	2.7	0.6	2.7325	$6.5\ 10^3$	$3\ 10^{-4}$
2.732055	−2.732050	2.732052	2.7	0.6	2.732046	$7.0\ 10^5$	$3\ 10^{-6}$

Table 7.2

Here D_K was computed with $\alpha = 0$ in Algorithm 6.1 and the value 1.0×10^{-5} was used as threshold on the singular values of $\Gamma = I - \gamma^{-2} X_\infty Y_\infty$ to chose between Algorithms 6.1 and 6.2. As a result, Algorithm 6.2 was activated for $\gamma = 2.732055$.

Inspection of Table 7.2 reveals a sharp degradation of internal stability as well as a rapid magnitude growth as $\gamma \to \gamma_{opt}$ with the usual algorithm. By contrast, the new approach totally neutralizes the near singularity in Γ so that stability remains robust and computations well-conditioned near γ_{opt}. Note that the results should be qualified by the fact that in this simple example, Algorithms 6.1-6.2 always returns a D_K for which $B_K = C_K = 0$. The resulting controller is therefore static with gain D_K.

■

Example 7.3 Consider the problem with data

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & -2 \end{pmatrix}; \; B_1 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}; \; B_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; \; C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}; \; C_2^T = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix};$$

$$D_{12}^T = (1\ ,\ 0\ ,\ 0); \qquad D_{21} = (0\ ,\ 0\ ,\ 1) \tag{7.2}$$

for which $\gamma_{opt} \approx 21.527873$. The experimental setup and parameters are as in Example 7.1 and the outcome is recorded in Table 7.4 which exactly parallels Table 7.2.

	Algorithms 6.1-6.2				Usual algorithm		
γ	D_K	$\|\mathcal{F}(G,K)\|_\infty$	$Mmax.$	$Smar.$	$\|\mathcal{F}(G,K)\|_\infty$	$Mmax.$	$Smar.$
40.0	23.2	25.3	3.9 10^1	1 10^{-3}	31.9	1.7 10^1	9 10^{-4}
25.0	22.2	22.8	3.7 10^1	8 10^{-4}	24.6	2.7 10^1	6 10^{-4}
22.0	21.6	21.7	3.7 10^1	6 10^{-4}	21.99	1.0 10^2	2 10^{-4}
21.6	21.54	21.56	3.7 10^1	6 10^{-4}	21.59	5.0 10^3	6 10^{-5}
21.53	21.528	21.528	3.7 10^1	6 10^{-4}	21.529	1.3 10^4	3 10^{-6}
21.528	21.5279	21.5279	3.7 10^1	6 10^{-4}	21.5278	2.2 10^5	2 10^{-7}
21.5279	21.52788	21.52788	3.7 10^1	6 10^{-4}	21.52788	1.2 10^6	4 10^{-8}
21.527874	21.527874	21.527874	3.7 10^1	6 10^{-4}	21.527874	1.2 10^7	2 10^{-9}

Table 7.4

The switching from Algorithm 6.1 to Algorithm 6.2 occurred early at $\gamma = 21.53$. These results confirm that the numerical condition of the computation of K_c deteriorates rapidly near γ_{opt} while the computation of generalized central controllers is little affected by this proximity. The transition between the two algorithms is smooth and even far from γ_{opt}, the closed-loop attenuation is better when using a feed-through term D_K. Note that the reduced-order "optimal" design was activated and valid relatively far from γ_{opt}. Hence it seems numerically safe to force small singular values of $I - \gamma^{-2} X_\infty Y_\infty$ to zero and perform optimal design with the remaining ones. Finally, the controller realization (A_K, B_K, C_K, D_K) computed by Algorithm 6.1 is continuous at γ_{opt}, becomes nonminimal, and reduces to the realization of the reduced-order optimal controller returned by Algorithm 6.2. This is illustrated by the following data drawn from the same experiment, except for the threshold which was chosen to postpone optimal design up to $\gamma = 21.527874$.

Sensitivity-reducing design:

$\underline{\gamma = 21.53\text{:}}$ $A_K = \begin{pmatrix} -21.23 & 15.71 & -0.001 \\ 10.83 & -8.14 & 0.005 \\ 0.0006 & -0.003 & -1.01 \end{pmatrix}$; $B_K = \begin{pmatrix} -21.73 \\ 12.45 \\ 0.0007 \end{pmatrix}$; $C_K^T = \begin{pmatrix} 19.35 \\ -15.42 \\ -0.0004 \end{pmatrix}$;

$\underline{\gamma = 21.5279\text{:}}$ $A_K = \begin{pmatrix} -21.23 & 15.71 & -0.0001 \\ 10.83 & -8.14 & 0.0006 \\ 0.0001 & -0.0003 & -1.02 \end{pmatrix}$; $B_K = \begin{pmatrix} -21.73 \\ 12.45 \\ 0.0001 \end{pmatrix}$; $C_K^T = \begin{pmatrix} 19.35 \\ -15.42 \\ -0.00006 \end{pmatrix}$.

Optimal design:

$\underline{\gamma = 21.527874\text{:}}$ $A_K = \begin{pmatrix} -21.23 & 15.71 \\ 10.83 & -8.14 \end{pmatrix}$; $B_K = \begin{pmatrix} -21.73 \\ 12.45 \end{pmatrix}$; $C_K^T = \begin{pmatrix} 19.35 \\ -15.42 \end{pmatrix}$.

■

8 Conclusion

Numerically reliable algorithms for the computation of H_∞ controllers near and at γ_{opt} have been presented. An instrumental innovation is the use of a controller feedthrough gain D_K to channel out the numerical sensitivity associated with $(I - \gamma^{-2}Y_\infty X_\infty)^{-1}$ near γ_{opt}. Undesirable side-effects of this ill-condition such as loss of accuracy, large magnitudes, performance degradation and/or weak internal stability are thus mostly eliminated. Comparative tests with existing algorithms for central controller computation have confirmed the validity of the method. In particular, closed-loop internal stability is substantially more robust with the new algorithm and the parameter magnitude is kept under control as γ approaches γ_{opt}.

Appendix A

Proof of Lemma 5.1: Observe that

$$X_\infty Y_\infty V_2 = \gamma_{opt}^2 V_2; \qquad Y_\infty X_\infty U_2 = \gamma_{opt}^2 U_2 \tag{A.1}$$

from the definition of U_2 and V_2. Pre- and post-multiply the first equation in (2.4) by $V_2^T Y_\infty$ and $Y_\infty V_2$, respectively, and use $X_\infty Y_\infty V_2 = \gamma_{opt}^2 V_2$ to obtain

$$V_2^T \left\{ AY_\infty + Y_\infty A^T + \gamma_{opt}^{-2} Y_\infty C_1^T C_1 Y_\infty + B_1 B_1^T \right\} V_2 \; - \; \gamma_{opt}^2 \, V_2^T B_2 B_2^T V_2 = 0.$$

Replacing the bracketed term by $Y_\infty C_2^T C_2 Y_\infty$ in virtue of the second equation in (2.4) then yields (5.4) and (5.5) follows by a duality argument.

Finally, consider the Hamiltonian matrix $H_0 = \begin{pmatrix} A & \gamma^{-2}B_1 B_1^T \\ -C_1^T C_1 & -A^T \end{pmatrix}$. From (2.4) it follows that

$$\begin{aligned} H_0 \begin{pmatrix} I \\ X_\infty \end{pmatrix} &= \begin{pmatrix} I \\ X_\infty \end{pmatrix} (A + \gamma^{-2} B_1 B_1^T X_\infty) - \begin{pmatrix} 0 \\ X_\infty B_2 B_2^T X_\infty \end{pmatrix}; \\ (I, -\gamma^{-2}Y_\infty) H_0 &= (A + \gamma^{-2} Y_\infty C_1^T C_1)(I, -\gamma^{-2}Y_\infty) + (0, \gamma^{-2} Y_\infty C_2^T C_2 Y_\infty). \end{aligned}$$

The expression $(I, -\gamma^{-2}Y_\infty) H_0 \begin{pmatrix} I \\ X_\infty \end{pmatrix}$ can then be evaluated in two different ways to obtain

$$\begin{aligned} &(I - \gamma^{-2} Y_\infty X_\infty)(A + \gamma^{-2} B_1 B_1^T X_\infty) + \gamma^{-2} Y_\infty X_\infty B_2 B_2^T X_\infty = \\ &\qquad (A + \gamma^{-2} Y_\infty C_1^T C_1)(I - \gamma^{-2} Y_\infty X_\infty) + \gamma^{-2} Y_\infty C_2^T C_2 Y_\infty X_\infty \end{aligned} \tag{A.2}$$

and pre- and post-multiplication of this equation by V_2^T and U_2, respectively, together with (A.1) establishes (5.6) for $\gamma = \gamma_{opt}$. The SVD's immediately follow from these identities.

■

References

[1] Bartels, R. H. and G. W. Stewart,"Solution of the Equation $AX + XB = C$," *Comm. Assoc. Comp. Mach.*, 15 (1972), pp. 820-826.

[2] Doyle, J.C., Glover, K., Khargonekar, P., and Francis, B., "State-Space Solutions to Standard H_2 and H_∞ Control Problems," *IEEE Trans. Aut. Contr.*, AC-34 (1989), pp. 831-847.

[3] Chang, B.C., X.P. Li, S.S. Banda, and H.H. Yeh," Design of an H_∞ Optimal Controller by Using DGKF's State-Space Formulas," *Proc. CDC*, 1990, pp. 2632-2633.

[4] Gahinet, P.M.,"On the Game Riccati Equations Arising in H_∞ Control Problems," submitted to *SIAM J. Contr. Opt.*, march 1991.

[5] Gahinet, P.M.,"A New Representation of H_∞ Suboptimal Controllers," submitted to ACC 92.

[6] Gahinet, P.M.,"On Theoretical and Numerical Limitations of the Classical State-Space Solution to H_∞ Control Problems," submitted to ACC 92.

[7] Glover, K. and J.C. Doyle,"State-space Formulae for all Stabilizing Controllers that Satisfy an H_∞-norm Bound and Relations to Risk Sensitivity," *Syst. Contr. Letters*, 11 (1988), pp. 167-172.

[8] Glover, K., D.J.N. Limebeer, J.C. Doyle, E.M. Kasenally, and M.G. Safonov,"A Characterization of all Solutions to the Four-Block General Distance Problem," *SIAM J. Contr. Opt.*, 29 (1991), pp. 283-324.

[9] Laub, A. J.,"A Schur Method for Solving Algebraic Riccati Equations," *IEEE Trans. Aut. Contr.*, AC-24 (1979), pp. 913-921.

[10] Parrott, S.,"On a Quotient Norm and the Sz.-Nagy-Foias Lifting Theorem," *J. Funct. Anal.*, 30 (1978), pp. 311-328.

[11] Safonov, M.G., D.J. Limebeer and R.Y. Chiang," Simplifying the H_∞ Theory via Loop-Shifting, Matrix-Pencil and Descriptor Concepts," *Int. J. Contr.*, 50 (1989), pp. 2467-2488.

[12] Scherer, C.,"The Riccati Inequality and State-Space H_∞-Optimal Control," *Ph.D. Dissertation*, Mathematisches Institut, Universitat Wurzburg, D-8700 Wurzburg, Germany, 1990.

[13] Smith, B.T., J.M. Boyle, J.J. Dongarra, B.S. Garbow, Y. Ikebe, V.C. Klema, and C.B. Moler, *Matrix Eigensystem Routines: EISPACK guide*, Springer Verlag, New York, 1972.

[14] Van Loan, C. F.,"How Near is a Stable Matrix to an Unstable Matrix?," *Contemp. Math.*, AMS, 47 (1985), pp. 465-478.

ISSN 0249 - 6399